Generalized Matrix Mechanics

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We propose a generalization of Heisenbergs' matrix mechanics based on many-index objects. It is shown that there exists a solution describing a harmonic oscillator and many-index objects lead to a generalization of spin algebra.

§1. Introduction

Until at the end of 19th century, it was usually believed that any experimental results could be explained with classical mechanics (CM). The black body radiation phenomena crushed this belief, and the concept of energy quanta was introduced by Planck in 1900 to overcome the difficulty. Afterwards quantum mechanics (QM) has been applied to far broader areas of physics with indisputable success. When one witnesses the triumphs, it is natural to ask the following questions.

1. Why does QM describe a microscopic world so successfully?
2. Does QM hold true without limit?
3. If there are limitations, how is QM modified beyond it?

Unfortunately we have no definite answers to them, although there are some conjectures. We expect that a generalization of CM and/or QM gives us a hint to answer the above questions. Hence it would be a meaningful task to construct a new mechanics based on CM and/or QM.

Nambu proposed a generalization of Hamiltonian dynamics by the extention of phase space based on the Liouville theorem and made a suggestion on its quantization.[1] The structure of this mechanics has been studied in the framework of the constrained system[2] and in a geometric and algebraic formulation.[3] There are several works towards quantization of Nambu mechanics.[3-8] This approach is quite interesting, but it is not a unique way to explore a new mechanics. There is a possibility to examine a generalization of QM directly, and we take a trial on this possibility.

In this paper, we propose a generalization of Heisenbergs' matrix mechanics based on many-index objects (we refer it as M-matrix).**) It is shown that there exists a solution describing a harmonic oscillator and many-index objects lead to a generalization of spin algebra. A conjecture on the operator formalism is also given.

This paper is organized as follows. In the next section, we review Heisenbergs' matrix mechanics and explore its generalization. We formulate (cubic) matrix mechanics based on three-index objects in §3. Section 4 is devoted to conclusions and discussion.

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**) Recently, Awata, Li, Minic and Yoneya have introduced many-index objects to quantize Nambu mechanics.[6] We will find that our definition of the triple-product among cubic matrices is different from theirs because we require a generalization of the Ritz rule in the phase factor, but not necessarily the associativity of the products.

§2. Matrix mechanics and generalization

2.1. *Heisenbergs' matrix mechanics*

We review Heisenbergs' matrix mechanics. For a closed physical system, the physical quantities are represented by hermitian square matrices such as

$$F_{mn}(t) = F_{mn} e^{i\Omega_{mn} t} = F_{mn} e^{\frac{i}{\hbar}(E_m - E_n)t}. \tag{2·1}$$

where the phase factor implies that the change in energy $E_m - E_n$ appears as radiation with the angular frequency Ω_{mn} and the hermiticity of $F_{mn}(t)$ is expressed by $F^*_{nm}(t) = F_{mn}(t)$. By a usual definition of the product of two square matrices $A_{mn}(t) = A_{mn} e^{i\Omega_{mn} t}$ and $B_{mn}(t) = B_{mn} e^{i\Omega_{mn} t}$ such that

$$(AB)_{mn}(t) \equiv \sum_k A_{mk}(t) B_{kn}(t) = \sum_k A_{mk} B_{kn} e^{i\Omega_{mn} t}, \tag{2·2}$$

the product $(AB)_{mn}(t)$ turns out to be the same form as (2·1) with the Ritz rule $\Omega_{mn} = \Omega_{mk} + \Omega_{kn}$. The time development of $F_{mn}(t)$ is expressed by the Heisenberg equation

$$\begin{aligned} \frac{d}{dt} F_{mn}(t) &= i\Omega_{mn} F_{mn} = \frac{i}{\hbar}(E_m - E_n) F_{mn}(t) \\ &= \frac{1}{i\hbar}((F(t)H)_{mn} - (HF(t))_{mn}) \equiv \frac{1}{i\hbar}[F(t), H]_{mn} \end{aligned} \tag{2·3}$$

where the Hamiltonian H is a diagonal matrix written by $H_{mn} \equiv E_m \delta_{mn}$.

Here we give a simple example of a harmonic oscillator whose variables are two hermitian matrices $\xi_{mn}(t) = \xi_{mn} e^{i\Omega_{mn} t}$ and $\eta_{mn}(t) = \eta_{mn} e^{i\Omega_{mn} t}$. The coefficients ξ_{mn} and η_{mn} are given by

$$\xi_{mn} = \sqrt{\frac{\hbar}{2m\Omega}} (\sigma^1)_{mn} \quad \text{and} \quad \eta_{mn}(t) = \sqrt{\frac{m\Omega\hbar}{2}} (\sigma^2)_{mn}, \tag{2·4}$$

respectively. Here the m in the square root represents a mass, the $(\sigma^a)_{mn}$ are Pauli matrices and $\Omega = \Omega_{21} (> 0)$. The $\xi_{mn}(t)$ and $\eta_{mn}(t)$ satisfy the following anticommutation relations,

$$\{\xi(t), \xi(t)\}_{mn} = \frac{\hbar}{m\Omega} \delta_{mn}, \quad \{\eta(t), \eta(t)\}_{mn} = m\Omega\hbar \delta_{mn}, \tag{2·5}$$

$$\{\xi(t), \eta(t)\}_{mn} = 0. \tag{2·6}$$

We have the equation of motion describing a harmonic oscillator

$$\frac{d}{dt} \xi_{mn}(t) = \frac{1}{i\hbar} [\xi, H]_{mn} = \frac{1}{m} \eta_{mn}(t), \tag{2·7}$$

$$\frac{d}{dt} \eta_{mn}(t) = \frac{1}{i\hbar} [\eta, H]_{mn} = -m\Omega^2 \xi_{mn}(t) \tag{2·8}$$

where the Hamiltonian H_{mn} is written by

$$H_{mn} = i\Omega \sum_k \xi_{mk}(t) \eta_{kn}(t) = -\frac{1}{2} \hbar \Omega (\sigma^3)_{mn}. \tag{2·9}$$

2.2. *Conjecture on M-matrix mechanics*

Let us extend a formulation described in the previous subsection to a system with M-matrix valued quantities whose variables are given by

$$F_{m_1 m_2 \cdots m_n}(t) = F_{m_1 m_2 \cdots m_n} e^{i\Omega_{m_1 m_2 \cdots m_n} t} \tag{2·10}$$

where the angular frequency $\Omega_{m_1 m_2 \cdots m_n}$ is written down by the use of antisymmetric quantities $\omega_{m_1 m_2 \cdots m_{n-1}}$ in the following

$$\Omega_{m_1 m_2 \cdots m_n} = \sum_{j=1}^{n} (-1)^{n-j} \omega_{m_1 \cdots m_{j-1} m_{j+1} \cdots m_n} \equiv (\partial\omega)_{m_1 m_2 \cdots m_n}. \tag{2·11}$$

Here we assume that there is a generalization of Bohrs' frequency condition*)

$$\Omega_{m_1 m_2 \cdots m_n} = \frac{1}{\hbar} \sum_{j=1}^{n} (-1)^{n-j} E_{m_1 \cdots m_{j-1} m_{j+1} \cdots m_n}. \tag{2·12}$$

The antisymmetric property is expressed by

$$\Omega_{m'_1 m'_2 \cdots m'_n} = \mathrm{sgn(P)} \Omega_{m_1 m_2 \cdots m_n}, \ \omega_{m'_1 m'_2 \cdots m'_{n-1}} = \mathrm{sgn(P)} \omega_{m_1 m_2 \cdots m_{n-1}} \tag{2·13}$$

where the sgn(P) denotes $+1$ and -1 for even and odd permutation among indices, respectively. The ∂ is regarded as a boundary operator which takes k-th antisymmetric objects into $(k+1)$-th objects, and this operation is nilpotent, i.e., $\partial^2(*) = 0$.[9] Hence a homology group can be constructed by a set of phase factor in M-matrices. The $\Omega_{m_1 m_2 \cdots m_n}$ are regarded as n-boundaries. We define the hermiticity of n-index object by $F_{m'_1 m'_2 \cdots m'_n}(t) = F^*_{m_1 m_2 \cdots m_n}(t)$ for odd permutation among subscripts. When we define a n-fold product among $F^{(a)}_{m_1 m_2 \cdots m_n}(t)$ $(a = 1, 2, ...)$ by

$$\begin{aligned}
(F^{(1)} \cdots F^{(n)})_{m_1 m_2 \cdots m_n}(t) &\equiv \sum_k F^{(1)}_{m_1 \cdots m_{n-1} k}(t) F^{(2)}_{m_1 \cdots m_{n-2} k m_n}(t) \cdots F^{(n)}_{k m_2 \cdots m_n}(t) \\
&= \sum_k (F^{(1)} \cdots F^{(n)})_{m_1 m_2 \cdots m_n} e^{i\Omega_{m_1 m_2 \cdots m_n} t},
\end{aligned} \tag{2·14}$$

the outcome has the same form as (2·10) with the relation $(\partial\Omega)_{m_1 m_2 \cdots m_{n+1}} = 0$, which is a generalization of the Ritz rule.

Next we discuss a time evolution of M-matrices $F^{(a)}_{m_1 m_2 \cdots m_n}(t)$. It is natural to have a conjecture that the equation of motion is given by

$$\begin{aligned}
\frac{d}{dt} F^{(a)}_{m_1 m_2 \cdots m_n}(t) &= i\Omega_{m_1 m_2 \cdots m_n} F^{(a)}_{m_1 m_2 \cdots m_n}(t) \\
&= \frac{i}{\hbar} \sum_{j=1}^{n} (-1)^{n-j} E_{m_1 \cdots m_{j-1} m_{j+1} \cdots m_n} F^{(a)}_{m_1 m_2 \cdots m_n}(t) \\
&= \frac{1}{i\hbar} [F^{(a)}(t), K^{(1)}, \cdots, K^{(n-2)}, H]_{m_1 m_2 \cdots m_n},
\end{aligned} \tag{2·15}$$

*) Here and hereafter we use a reduced Planck constant $\hbar = \frac{h}{2\pi}$ as a unit of action with an expectation that M-matrix mechanics contains QM as a limit and shares a same physical constant.

where the $K^{(1)}, \cdots, K^{(n-2)}$ and H are time-independent n-index objects called *Hamiltonians* and they are functions of n-index variables $F^{(a)}_{m_1 m_2 \cdots m_n}(t)$. In the above equation, the n-fold commutator is defined by

$$[F^{(a_1)}, F^{(a_2)}, \cdots, F^{(a_n)}]_{m_1 m_2 \cdots m_n} \equiv \sum \mathrm{sgn}(\mathrm{P})(F^{(a'_1)} F^{(a'_2)} \cdots F^{(a'_n)})_{m_1 m_2 \cdots m_n} \tag{2·16}$$

where the summation is done for all permutations among superscripts. The equation (2·15) is regarded as a generalization of Heisenberg equation. In the next section, we study three-index objects and their dynamics, and write down an explicit form for Hamiltonians.

We give a comment on a set of n-index object. We find that the $(n+1) \times (n+1) \times \cdots \times (n+1)$ matrices defined by $J^{(a)}_{m_1 m_2 \cdots m_n} \equiv -i\hbar \varepsilon_{a m_1 m_2 \cdots m_n}$ satisfy the following interesting algebra

$$[J^{(a_1)}, J^{(a_2)}, \cdots, J^{(a_n)}]_{m_1 m_2 \cdots m_n} = (-i)^{n+1} \hbar^{n-1} \varepsilon_{a_{n+1} a_1 a_2 \cdots a_n} J^{(a_{n+1})}_{m_1 m_2 \cdots m_n}. \tag{2·17}$$

This algebra is a generalization of spin algebra ($su(2)$ algebra) and is equivalent to a special case of M-algebra discussed in 5.

2.3. *Relation to classical dymanics*

Before we study a cubic matrix, we discuss a structure of classical dynamics from a viewpoint of matrix mechanics. The physical variable $F(t)$ in CM is regarded as a linear combination of one-index object (1×1 matrix) such that

$$F(t) = \sum_n F_n e^{i\Omega_n t} \tag{2·18}$$

where $F^*_n = F_{-n}$ because $F(t)$ should be a real quantity, and the angular frequency Ω_n is a multiple of a basic one ω, i.e., $\Omega_n = n\omega$. Under a guidance of Bohrs' correspondence principle and frequency condition, we obtain a relation between ω and the Hamiltonian H such that

$$\omega = \frac{\Omega_{\Delta n}}{\Delta n} = \lim_{\frac{\hbar \Delta n}{n} \to 0} \frac{\Omega_{n+\Delta n n}}{\Delta n} = \lim_{\frac{\hbar \Delta n}{n} \to 0} \frac{E_{n+\Delta n} - E_n}{\hbar \Delta n} = \frac{dE}{dJ} = \frac{\partial H}{\partial J} \tag{2·19}$$

where the J is action variable and we use $J = \oint p dq = hn$ (Bohr-Sommerfeld quantization condition). The equation of motion for $F(t)$ is written by

$$\frac{d}{dt} F(t) = \sum_n i n \omega F_n e^{i\Omega_n t} = \frac{\partial F(t)}{\partial(\omega t)} \frac{\partial H}{\partial J} = \{F(t), H\}_{\mathrm{PB}} \tag{2·20}$$

where the $\{*, *\}_{\mathrm{PB}}$ is the Poisson bracket and we use a fact that J is a canonical conjugate to angle variable ωt. The equation (2·20) is just Hamiltons' canonical equation.

§3. Cubic matrix mechanics

3.1. *Cubic matrix*

We consider three-index object (cubic matrix) given by

$$C_{lmn}(t) = C_{lmn} e^{i\Omega_{lmn} t} \tag{3·21}$$

where the C_{lmn} have a cyclic symmetry, i.e., $C_{lmn} = C_{mnl} = C_{nlm}$, and the angular frequency Ω_{lmn} has a following form

$$\Omega_{lmn} = \omega_{lm} - \omega_{ln} + \omega_{mn} \equiv (\partial\omega)_{lmn}, \quad \omega_{ml} = -\omega_{lm}. \tag{3·22}$$

The Ω_{lmn} shows following properties,

$$\Omega_{l'm'n'} = \mathrm{sgn(P)}\Omega_{lmn}, \tag{3·23}$$
$$(\partial\Omega)_{lmnk} \equiv \Omega_{lmn} - \Omega_{lmk} + \Omega_{lnk} - \Omega_{mnk} = 0. \tag{3·24}$$

The relations (3·22) and (3·24) show that the Ω_{lmn} are 3-boundaries when the ∂ is regarded as a boundary operator. We define the hermiticity of cubic matrix by $C_{l'm'n'}(t) = C^*_{lmn}(t)$ for odd permutation among indices. For a hermitian cubic matrix, there are following relations,

$$C_{lmn}(t) = C_{mnl}(t) = C_{nlm}(t) = C^*_{mln}(t) = C^*_{lnm}(t) = C^*_{nml}(t). \tag{3·25}$$

When we define a triple-product among cubic matrices $C_{lmn}(t)$, $D_{lmn}(t)$ and $E_{lmn}(t)$ by

$$(C(t)D(t)E(t))_{lmn} \equiv \sum_k C_{lmk}(t)D_{lkn}(t)E_{kmn}(t) = (CDE)_{lmn} e^{i\Omega_{lmn} t}, \tag{3·26}$$

the product takes the same form as (3·21) with the relation (3·24). Note that the product is, in general, not commutative and not associative, e.g., $(CDE)_{lmn} \neq (DCE)_{lmn}$ and $(AB(CDE))_{lmn} \neq (A(BCD)E)_{lmn} \neq ((ABC)DE)_{lmn}$, respectively. Taking a hermitian conjugation of products for hermitian cubic matrices, we obtain the relations such that

$$\begin{aligned}(C(t)D(t)E(t))_{lmn} &= (E(t)D(t)C(t))^*_{nml} = (C(t)E(t)D(t))^*_{mln} \\ &= (D(t)C(t)E(t))^*_{lnm} = (D(t)E(t)C(t))_{nlm} = (E(t)C(t)D(t))_{mnl}.\end{aligned} \tag{3·27}$$

The triple-commutator and anticommutator are defined by

$$\begin{aligned}[C(t), D(t), E(t)]_{lmn} \equiv (&C(t)D(t)E(t) + D(t)E(t)C(t) + E(t)C(t)D(t) \\ &- D(t)C(t)E(t) - C(t)E(t)D(t) - E(t)D(t)C(t))_{lmn}\end{aligned} \tag{3·28}$$

and

$$\begin{aligned}\{C(t), D(t), E(t)\}_{lmn} \equiv (&C(t)D(t)E(t) + D(t)E(t)C(t) + E(t)C(t)D(t) \\ &+ D(t)C(t)E(t) + C(t)E(t)D(t) + E(t)D(t)C(t))_{lmn},\end{aligned} \tag{3·29}$$

respectively. By the above definition, we have a relation such as

$$[A^{(a')}(t), A^{(b')}(t), A^{(c')}(t)]_{lmn} = \mathrm{sgn(P)}[A^{(a)}(t), A^{(b)}(t), A^{(c)}(t)]_{lmn}. \tag{3·30}$$

If $C_{lmn}(t)$, $D_{lmn}(t)$ and $E_{lmn}(t)$ are hermitian matrices, the $[C(t), D(t), E(t)]_{lmn}$ and $\{C(t), D(t), E(t)\}_{lmn}$ are also hermitian cubic matrices.

3.2. *Dynamics*

The cyclically-symmetric cubic matrices $C^{(a)}_{lmn}(t)$ yield a generalization of Heisenberg equation

$$\frac{d}{dt}C^{(a)}_{lmn}(t) = i\Omega_{lmn}C^{(a)}_{lmn}(t) = \frac{1}{i\hbar}[C^{(a)}(t), K, H]_{lmn}, \tag{3·31}$$

where K and H are time-independent 3-index objects. A possible form of K and H are given by

$$K_{lmn} = I_{lm}{}^{n} + I^{l}{}_{mn} + I_{l}{}^{m}{}_{n} \equiv I_{lmn}, \tag{3·32}$$

$$H_{lmn} = \frac{1}{2}\hbar\omega_{mn}I_{lm}{}^{n} + \frac{1}{2}\hbar\omega_{nl}I^{l}{}_{mn} + \frac{1}{2}\hbar\omega_{lm}I_{l}{}^{m}{}_{n} \tag{3·33}$$

where the $I_{lm}{}^{n}$, $I_{l}{}^{m}{}_{n}$ and $I^{l}{}_{mn}$ are defined by

$$I_{lm}{}^{n} \equiv \delta_{lm}(1-\delta_{nl}), \quad I_{l}{}^{m}{}_{n} \equiv \delta_{ln}(1-\delta_{mn}), \quad I^{l}{}_{mn} \equiv \delta_{mn}(1-\delta_{lm}). \tag{3·34}$$

Our mechanics could not be interpreted as a quantum realization of Nambu mechanics because our triple-commutator, in general, does not satisfy the conditions such as the derivation rule (which is a counterpart of Leibniz rule in differential calculus) and a generalization of Jacobi identity called *a fundamental identity*, both of which the Nambu-Poisson bracket possesses. As an exceptional case, the derivation rule and the fundamental identity hold for the triple-commutator including the Hamiltonians K and H, e.g.,

$$\begin{aligned}\frac{d}{dt}(C(t)D(t)E(t))_{lmn} &= \left(\frac{dC(t)}{dt}D(t)E(t)\right)_{lmn} + \left(C(t)\frac{dD(t)}{dt}E(t)\right)_{lmn} \\ &\quad + \left(C(t)D(t)\frac{dE(t)}{dt}\right)_{lmn} \\ &= i\Omega_{lmn}(C(t)D(t)E(t))_{lmn} \\ &= ([C(t),K,H]D(t)E(t))_{lmn} + (C(t)[D(t),K,H]E(t))_{lmn} \\ &\quad + (C(t)D(t)[E(t),K,H])_{lmn} \\ &= [C(t)D(t)E(t),K,H]_{lmn}\end{aligned} \tag{3·35}$$

for $(CDE)_{llm} = (CDE)_{lml} = (CDE)_{mll}$ and

$$\begin{aligned}&[[C(t),D(t),E(t)],K,H]_{lmn} = [[C(t),K,H],D(t),E(t)]_{lmn} \\ &\quad + [C(t),[D(t),K,H],E(t)]_{lmn} + [C(t),D(t),[E(t),K,H]]_{lmn}.\end{aligned} \tag{3·36}$$

In this way, our description of time development is consistent for cyclically-symmetric matrices.

3.3. *Examples*

We study a simple example of a harmonic oscillator whose variables are two hermitian $3\times3\times3$ matrices $\xi_{lmn}(t) = \xi_{lmn}e^{i\Omega_{lmn}t}$ and $\eta_{lmn}(t) = \eta_{lmn}e^{i\Omega_{lmn}t}$. The coefficients ξ_{lmn} and η_{lmn} are given by

$$\xi_{lmn} = -\sqrt{\frac{\hbar}{2m\Omega}}\frac{\Omega_{lmn}}{\Omega}\varepsilon_{lmn}, \quad \eta_{lmn}(t) = \frac{1}{i}\sqrt{\frac{m\Omega\hbar}{2}}\varepsilon_{lmn} \tag{3·37}$$

where the m in the square root represents a mass and $\Omega = \Omega_{321}(> 0)$. The $\xi_{lmn}(t)$ and $\eta_{lmn}(t)$ satisfy the following relations,

$$(I\xi^2)_{lmn} = \frac{\hbar}{2m\Omega} I_{lm}{}^{n}, \ (\xi^2 I)_{lmn} = \frac{\hbar}{2m\Omega} I^{l}{}_{mn}, \ (\xi I \xi)_{lmn} = \frac{\hbar}{2m\Omega} I_{l}{}^{m}{}_{n}, \tag{3·38}$$

$$(I\eta^2)_{lmn} = \frac{\hbar m\Omega}{2} I_{lm}{}^{n}, \ (\eta^2 I)_{lmn} = \frac{\hbar m\Omega}{2} I^{l}{}_{mn}, \ (\eta I \eta)_{lmn} = \frac{\hbar m\Omega}{2} I_{l}{}^{m}{}_{n}, \tag{3·39}$$

$$\begin{aligned}(I\xi\eta)_{lmn} + (I\eta\xi)_{lmn} &= (\xi\eta I)_{lmn} + (\eta\xi I)_{lmn} \\ &= (\xi I\eta)_{lmn} + (\eta I\xi)_{lmn} = 0,\end{aligned} \tag{3·40}$$

$$(\xi^3)_{lmn} = (\xi^2\eta)_{lmn} = \cdots = (\eta^2\xi)_{lmn} = (\eta^3)_{lmn} = 0 \tag{3·41}$$

where $I = I_{lmn}$. We have the equation of motion describing a harmonic oscillator

$$\frac{d}{dt}\xi_{lmn}(t) = \frac{1}{i\hbar}[\xi, K, H]_{lmn} = \frac{1}{m}\eta_{lmn}(t), \tag{3·42}$$

$$\frac{d}{dt}\eta_{lmn}(t) = \frac{1}{i\hbar}[\eta, K, H]_{lmn} = -m\Omega^2\xi_{lmn}(t) \tag{3·43}$$

where the K and H are the Hamiltonians. Here two kinds of solutions for the K and H are listed. One is a same set as given by (3·32) and (3·33), and it is rewritten down as

$$\begin{aligned}K_{lmn} &= \frac{m\Omega}{\hbar}((I\xi^2)_{lmn} + (\xi^2 I)_{lmn} + (\xi I\xi)_{lmn}) \\ &\quad + \frac{1}{m\Omega\hbar}((I\eta^2)_{lmn} + (\eta^2 I)_{lmn} + (\eta I\eta)_{lmn}),\end{aligned} \tag{3·44}$$

$$\begin{aligned}H_{lmn} &= \frac{1}{2}m\Omega(\omega_{mn}(I\xi^2)_{lmn} + \omega_{nl}(\xi^2 I)_{lmn} + \omega_{lm}(\xi I\xi)_{lmn}) \\ &\quad + \frac{1}{2m\Omega}(\omega_{mn}(I\eta^2)_{lmn} + \omega_{nl}(\eta^2 I)_{lmn} + \omega_{lm}(\eta I\eta)_{lmn})\end{aligned} \tag{3·45}$$

by the use of $\xi_{lmn}(t)$ and $\eta_{lmn}(t)$. The other one is written down as

$$\begin{aligned}K_{lmn} &= \frac{2i}{\hbar}((I\xi\eta)_{lmn} + (\xi\eta I)_{lmn} + (\eta I\xi)_{lmn}) \\ &= I^{(3)}_{lm}{}^{n} + I^{(3)l}{}_{mn} + I^{(3)}_{l}{}^{m}{}_{n},\end{aligned} \tag{3·46}$$

$$\begin{aligned}H_{lmn} &= i\omega_{mn}(I\xi\eta)_{lmn} + i\omega_{nl}(\xi\eta I)_{lmn} + i\omega_{lm}(\eta I\xi)_{lmn} \\ &= \frac{1}{2}\hbar\omega_{mn}I^{(3)}_{lm}{}^{n} + \frac{1}{2}\hbar\omega_{nl}I^{(3)l}{}_{mn} + \frac{1}{2}\hbar\omega_{lm}I^{(3)}_{l}{}^{m}{}_{n}.\end{aligned} \tag{3·47}$$

Here the $I^{(3)}_{lm}{}^{n}$, $I^{(3)l}{}_{mn}$ and $I^{(3)}_{l}{}^{m}{}_{n}$ are defined by

$$I^{(3)}_{lm}{}^{n} \equiv \delta_{lm}\varepsilon_{mn}, \quad I^{(3)l}{}_{mn} \equiv \delta_{mn}\varepsilon_{nl}, \quad I^{(3)}_{l}{}^{m}{}_{n} \equiv \delta_{ln}\varepsilon_{lm} \tag{3·48}$$

where $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = -\varepsilon_{21} = -\varepsilon_{32} = -\varepsilon_{13} = 1$.

3.4. *Operator formalism*

We have discussed a generalization of QM by using M-matrix. The mechanics has an interesting algebraic structure, but the formalism is not so practical because it is only applicable to the stationary system. From experience, we expect that the operator formalism must be useful to handle problems in a wider class of physical system. By analogy with QM, we discuss the operator formalism of cubic matrix mechanics. First we take following basic assumptions.

1. For a given physical system, there exist triplet of state vectors $|m_1; \mathrm{P}_{m_1m_2m_3}\rangle$, $|m_2; \mathrm{P}_{m_1m_2m_3}\rangle$, $|m_3; \mathrm{P}_{m_1m_2m_3}\rangle$ depending on both quantum numbers m_i, (e.g., m_i are l, m or n) and their ordering. Here the ordering is represented by the permutation (denoted by $\mathrm{P}_{m_1m_2m_3}$) for a standard one, (e.g., $m_1 = l, m_2 = m, m_3 = n$.
2. For every physical observable, there is a ono-to-one correspondence to a linear operator C.

Under the above assumptions, it is natural to identify the cubic matrix element C_{lmn} with $C|l; \mathrm{P}_{lmn}\rangle|m; \mathrm{P}_{lmn}\rangle|n; \mathrm{P}_{lmn}\rangle$. In general, the $C_{m_1m_2m_3}$ is identified with with $C|m_1; \mathrm{P}_{m_1m_2m_3}\rangle|m_2; \mathrm{P}_{m_1m_2m_3}\rangle|m_3; \mathrm{P}_{m_1m_2m_3}\rangle$. By the use of (3·31), the following equations of motion for the states are derived

$$i\hbar\frac{d}{dt}|l; \mathrm{P}_{lmn}\rangle = [K, H]|l; \mathrm{P}_{lmn}\rangle, \quad i\hbar\frac{d}{dt}|m; \mathrm{P}_{lmn}\rangle = [K, H]|m; \mathrm{P}_{lmn}\rangle,$$
$$i\hbar\frac{d}{dt}|n; \mathrm{P}_{lmn}\rangle = [K, H]|n; \mathrm{P}_{lmn}\rangle \tag{3·49}$$

where the $[K, H]$ is a commutator of the Hamiltonian operators K and H, e.g., the $[K, H]$ in the third equation corresponds to $\sum_k (K_{lkn}H_{kmn} - H_{lkn}K_{kmn})$ in cubic matrix mechanics. The above equations (3·49) are regarded as a generalization of Schrödinger equation. By the use of (3·32) and (3·33), the time evolution of state vectors are given by

$$|l; \mathrm{P}_{lmn}\rangle = \exp\left(\frac{i}{2}(\omega_{nl} + \omega_{lm})t\right)|l; \mathrm{P}_{lmn}\rangle_0,$$
$$|m; \mathrm{P}_{lmn}\rangle = \exp\left(\frac{i}{2}(\omega_{lm} + \omega_{mn})t\right)|m; \mathrm{P}_{lmn}\rangle_0,$$
$$|n; \mathrm{P}_{lmn}\rangle = \exp\left(\frac{i}{2}(\omega_{mn} + \omega_{nl})t\right)|n; \mathrm{P}_{lmn}\rangle_0 \tag{3·50}$$

where the 0 stands for the values at an initial time. In the same way, the time development of state vectors for the matrix element C_{mln} are given by

$$|l; \mathrm{P}_{mln}\rangle = \exp\left(\frac{i}{2}(\omega_{ml} + \omega_{ln})t\right)|l; \mathrm{P}_{mln}\rangle_0,$$
$$|m; \mathrm{P}_{mln}\rangle = \exp\left(\frac{i}{2}(\omega_{nm} + \omega_{ml})t\right)|m; \mathrm{P}_{mln}\rangle_0,$$
$$|n; \mathrm{P}_{mln}\rangle = \exp\left(\frac{i}{2}(\omega_{ln} + \omega_{nm})t\right)|n; \mathrm{P}_{mln}\rangle_0. \tag{3·51}$$

We can identify $|l; \mathrm{P}_{mln}\rangle$ with a complex conjugate of $|l; \mathrm{P}_{lmn}\rangle$ from (3·50) and (3·51), and find that this identification is consistent with the relations (3·25).

§4. Conclusions and discussion

We have proposed a generalization of Heisenbergs' matrix mechanics based on many-index objects. It has been shown that there exists a solution describing a harmonic oscillator, e.g., the three-index objects $(\xi_{lmn}(t), \eta_{lmn}(t))$ defined by (3·37) satisfy the equations (3·42) and (3·43), and many-index objects lead to a generalization of spin algebra, e.g., the $4 \times 4 \times 4$ matrices defined by $J^{(a)}_{lmn} \equiv -i\hbar\varepsilon_{almn}$ satisfy the following algebra

$$[J^{(a)}, J^{(b)}, J^{(c)}]_{lmn} = \hbar^2 \varepsilon_{abcd} J^{(d)}_{lmn} \tag{4·52}$$

where a, b, c, d, l, m, n are integers from 1 to 4. We have given a conjecture on the operator formalism. The basic equations are given by (3·49).

Finally we give comments on the questions raised in the introduction.

For the first question 'Why does QM describe a microscopic world so successfully?', *the simplicity or variety of structure* in mechanics would be a key word. QM might make a special position in a set of M-matrix mechanics. For example, matrix mechanics with many-index objects could be reduced to Heisenbergs' matrix mechanics or a trivial theory by a change of variables. It is important to make clear a whole structure of M-matrix mechanics and find a relation between them.

For the second question 'Does QM hold true without limit?', there is a suggestion that QM should be modified near the Planck scale based on the problem on information loss in black hole.[10] This problem is deeply rooted in the obstacle to the quantization of gravity. The superstring theory and/or M-theory are the most promising theory including quantum gravity. In fact, the problem on the counting of entropy is solved for a class of (near)-extremal black holes in the superstring theory.[11]

For the third question 'If there are limitations, how is QM modified beyond it?', if elementary objects in nature are not point particles but some extented objects, a right way to arrive a final theory must be to construct a theory based on a (new) mechanics appropriate to fundamental constituents. Or there is a possibility that the superstring theory and/or M-theory build in a new mechanics. It would be worthwhile to explore a generalization of QM to approach a fundamental theory of nature from every possible aspects.[*]

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[*] Several different ideas have been proposed for a generalization of QM.[12]

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